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On the principle of exchange of stabilities for the magnetohydrodynamic thermal stability problem in completely confined fluids

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The principle of exchange of stabilities (exchange principle) for the thermal stability problem has been proved by Pellew & Southwell (1940) for fluids bounded by two infinite, horizontal parallel planes. Chandrasekhar (1952) discussed the establishment of the exchange principle for the same geometry when the fluid is an electrical conductor and when an arbitrary oriented, uniform, external magnetic field is applied in the vertical direction.

In this paper, the exchange principle is examined for fluids completely confined in an arbitrary region with rigid bounding surfaces that are good electrical conductors with respect to the fluid. The uniform magnetic field is applied in an arbitrary direction. A generalized thermal boundary condition is imposed which includes the fixed temperature and prescribed heat-flux conditions as special cases.

If no magnetic field is applied to the fluid, the present work reduces to a generalization (for completely confined fluids) of the Pellew & Southwell proof of the exchange principle. In the magnetohydrodynamic (MHD) thermal stability problem, the exchange principle is found to be valid if the total kinetic energy associated with an arbitrary disturbance is greater than or equal to its total magnetic energy. In a special case it is demonstrated that a sufficient condition which will establish the exchange principle is $k \leqslant \eta$, where k is the fluid thermal diffusivity and η is the fluid electrical resistivity.

1. Introduction

In a certain class of stability problems, the unsteady terms may be eliminated from the governing linearized disturbance equations. A necessary condition for this to be valid is given by the principle of exchange of stabilities. In simple terms the principle expresses the fact that while temporally oscillating disturbances are not excluded, they will in fact be damped out. Notable mathematical simplification results from the use of the principle, since the transition from stability to instability occurs via a marginal stationary state. This state is characterized by the vanishing of both the real and imaginary parts of the complex time eigenvalue associated with the disturbance.

The principle of exchange of stabilities was used by Rayleigh (1916) in the first theoretical investigation of the thermal stability problem for a layer of fluid

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bounded by two infinite horizontal planes. Pellew & Southwell (1940) formally established the principle for Rayleigh's geometrical configuration. Yih (1959) extended the proof for fluids bounded either by infinitely long, insulated vertical tubes or by vertical cylinders with insulated side walls and fixed-temperature rigid horizontal planes. Ostrach & Pnueli (1963), Velte (1964), and Weinbaum (1964) treat the thermal stability problem for more arbitrary regions. Sparrow, Goldstein & Jonsson (1964) consider the horizontal layer configuration with a more general thermal boundary condition on the horizontal bounding planes. In all of these works the exchange principle is accepted and used without proof.

The thermal stability problem has been extended to include the effect of a uniform, applied magnetic field on an electrically conducting fluid. Thompson (1951) discussed the establishment of the exchange principle for the MHD thermal stability problem, but his analysis is somewhat limited. He treats an inviscid layer of fluid bounded by two infinite horizontal planes. The uniform, applied magnetic field is alined with the direction of the non-magnetic body force. Chandrasekhar (1952) examined the same configuration as Thompson but his paper goes somewhat further by including the effect of viscosity in the governing equations. Chandrasekhar begins by assuming a particularly simple solution to the governing time-dependent disturbance equations. He then demonstrates that the principle of exchange of stabilities will be valid if the fluid properties satisfy a certain necessary and sufficient condition, i.e.

$$k \leqslant \eta, \tag{1}$$

where k is the fluid thermal diffusivity and η is the fluid electrical resistivity. However, the assumed solution modes satisfy a limited set of hydrodynamic and thermal boundary conditions. For the special case treated by Chandrasekhar, the horizontal bounding planes must be isothermal, free surfaces. Moreover, the gradient of the normal component of the induced magnetic field must vanish on these bounding planes. This artificial magnetic-field boundary condition would prove difficult to maintain in any realistic situation.

In an effort to establish the exchange principle for more general and more realistic boundary conditions on the horizontal planes so that the deficiencies in the above method are eliminated, Chandrasekhar (1952) also proceeded in an alternate manner. He extended the proof of the exchange principle developed by Pellew & Southwell (1940) for the thermal stability problem to include the effects of fluid electrical conductivity and a uniform, applied magnetic field acting in the vertical direction.

Using this method, a variety of hydrodynamic and thermal boundary conditions may be specified on the horizontal bounding planes. These bounding surfaces must be good electrical conductors in comparison with the fluid in order to enforce certain necessary boundary conditions on the induced magnetic field. Chandrasekhar surmized the following from his alternate approach: It appears likely that in order for the exchange principle to be established in the MHD thermal stability problem, the total kinetic energy associated with a disturbance must be greater than or equal to the total magnetic energy. Chandrasekhar found that the method of Pellew & Southwell was not quite strong enough to demonstrate rigorously this supposition.

In what follows, the exchange principle is examined for an electrically conducting fluid that is completely confined within an arbitrary region whose bounding surfaces are rigid walls. A generalized thermal boundary condition may be specified on the bounding surfaces, and these surfaces must be good electrical conductors relative to the fluid. Previous analyses required the uniform, applied magnetic field to be alined with the non-magnetic body force. In the present paper, the uniform field may be applied in an arbitrary direction.

2. The governing equations

Consider an arbitrary, completely enclosed region within which a quasi-incompressible (Boussinesq) fluid with a positive coefficient of thermal expansion is confined. By imposition of certain prescribed thermal boundary conditions on the rigid bounding walls, a constant temperature gradient is maintained parallel to the body force acting on the fluid. If the temperature gradient is in the direction of the body force, then a potentially unstable, 'top-heavy' arrangement results. The object is to find the critical temperature gradient above which there is a tendency for the fluid to move in an attempt to remedy the unstable situation.

The MHD thermal stability problem is associated with the solution of a set of linearized disturbance equations describing the interaction between perturbations in the magnetic field, modifications of the initial temperature distribution, and fluid motions. The non-magnetic momentum equation is modified to take into account the Lorentz force, while the continuity equation and the energy equation are exactly as in the non-magnetic analysis. To these equations one must adjoin Maxwell's equations (incorporating the well-known MHD approximation that neglects displacement currents and excess charge density in the bulk of the fluid) and Ohm's law (in which Hall effects are neglected). Combining Maxwell's equations and Ohm's law in the usual manner so as to eliminate the electric field and the current density yields the so-called magnetic equation. Thus the MHD thermal stability problem is associated with the solution of the eigenvalue problem posed by†

$$\operatorname{div}\mathbf{u} = 0, \tag{2}$$

$$\frac{1}{P_1}\frac{\partial \mathbf{u}}{\partial t} = -Ra\,\theta\hat{k} - \frac{1}{P_1}\operatorname{grad}p - \operatorname{curl}\operatorname{curl}\mathbf{u} + Q\,(\operatorname{curl}\mathbf{h})\times\hat{l}, \tag{3}$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u}.\hat{k} = \nabla^2\theta, \tag{4}$$

$$\frac{P_2}{P_1}\frac{\partial \mathbf{h}}{\partial t} - \operatorname{curl}(\mathbf{u}\times\hat{l}) = -\operatorname{curl}\operatorname{curl}\mathbf{h}, \tag{5}$$

$$\operatorname{div}\mathbf{h} = 0, \tag{6}$$

where Chandrasekhar's notation has been used. In the above dimensionless governing equations, $\mathbf{u}$ is the disturbance velocity, θ is the temperature modifica-

† The assumptions used in deriving these equations are discussed by Chandrasekhar (1961).

tion of the initial linear profile, p is the pressure deviation from the initial distribution, and $\mathbf{h}$ is the induced magnetic field. The direction of the uniform, applied magnetic field is given by the unit vector $\hat{l}$, and the direction of the non-magnetic body force is given by the unit vector $\hat{k}$. The equations were cast in dimensionless form by introducing the following characteristic quantities: a characteristic length d, the maximum dimension of the region in the non-magnetic body-force direction; a characteristic velocity k/d; a characteristic pressure $\rho k^2/d^2$, where ρ is the mean fluid density; a characteristic temperature difference βd, where β is the constant temperature gradient; a characteristic time d^2/k; and a characteristic induced magnetic field $(k/\eta) H_0$, where H_0 is the magnitude of the applied field.

Four parameters appear in the equations: Prandtl number, $P_1 = \nu/k$, where ν is the fluid kinematic viscosity; magnetic Prandtl number, $P_2 = \nu/\eta$; magneto-viscous parameter (square of Hartmann number), $Q = \mu H_0^2 d^2/4\pi\rho\nu\eta$, where μ is the fluid magnetic permeability; and Rayleigh number, $Ra = -g\alpha\beta d^4/k\nu$, where g is the non-magnetic body force per unit mass in the $-\hat{k}(-z)$-direction and α is the fluid coefficient of thermal expansion. The Rayleigh number is a positive quantity when β, the temperature gradient, is negative (directed *with* the body force).

3. The boundary conditions

Associated with these equations are a set of homogeneous, time-independent boundary conditions. Since this analysis is limited to regions completely confined by rigid walls, the no-slip and no-through-flow conditions require that all velocity components vanish on the bounding surfaces. Thus

$$\mathbf{u} = 0 \tag{B.C. 1}$$

on rigid walls for all times.

A general thermal convective-radiative exchange between the fluid and its surrounding environment may be prescribed at a surface. In this situation, Sparrow, Goldstein & Jonsson (1964) show that the temperature perturbation θ must satisfy

$$\nabla\theta . \hat{n} + Bi\,\theta = 0 \tag{B.C. 2}$$

on the surface for all times. Here the Biot number $Bi = hd/\kappa$ has been introduced. h is the positive, time-independent conductance coefficient from the surface to the environment and κ is the fluid thermal conductivity (note that the Biot number is always positive). For large values of the Biot number, the surface has the character of the fixed temperature surface ($\theta = 0$). For small values of the Biot number, the surface boundary condition approaches that for prescribed heat flux ($\nabla\theta . \hat{n} = 0$).

Recall that the initial temperature distribution is fixed in this analysis to be linear in the direction of the body force. This is the basic cause of the instability under consideration. Hence care must be exercised to insure that the thermal boundary conditions are prescribed so as to be consistent with this specification. The shape of the region under consideration will be a determining factor in the choice of these conditions.

The boundary condition to be imposed on the induced magnetic field requires some discussion. It is known that across any interface the tangential component of the electric field, $\mathbf{e}$, must be continuous. Thus at a bounding surface one has

$$\hat{n} \times (\mathbf{e} - \mathbf{e}_w) = 0,$$

where $\mathbf{e}_w$ represents the induced electric field in the wall. Since $\mathbf{u} = 0$ (B.C. 1) on all rigid walls then, using Ohm's law, one can rewrite the above condition as

$$\hat{n} \times (\operatorname{curl} \mathbf{h} - (\sigma/\sigma_w)(\operatorname{curl} h)_w) = 0,$$

where σ and σ_w are the electrical conductivities of the fluid and of the wall, respectively. If the wall's electrical conductivity is large in comparison with the electrical conductivity of the fluid, then as $\sigma/\sigma_w \to 0$,

$$\hat{n} \times \operatorname{curl} \mathbf{h} = 0 \qquad \text{(B.C. 3)}$$

on all bounding surfaces for all times. This boundary condition is the only condition needed on the induced magnetic field in the proof of the exchange principle that follows.

Note for the non-magnetic case one of the boundaries could be a free surface. For a stationary free-surface there can be no through-flow so that

$$\mathbf{u} \cdot \hat{n} = 0, \qquad \text{(B.C. } 4a\text{)}$$

where $\hat{n}$ is an outward normal to a surface element. Also, no tangential stresses are permitted on free surfaces. It can be shown that for planar, free surfaces the two conditions stated above together with the solenoidal property of the velocity field result in vorticity lines lying perpendicular to such surfaces, i.e.

$$\hat{n} \times \operatorname{curl} \mathbf{u} = 0. \qquad \text{(B.C. } 4b\text{)}$$

4. Proof

Consider the time-dependent variables $\mathbf{u}$, θ, p, and $\mathbf{h}$ to have the form

$$\mathbf{u} = \mathbf{U}(x,y,z)\,e^{nt}, \quad p = \Pi(x,y,z)\,e^{nt},$$
$$\theta = \Theta(x,y,z)\,e^{nt}, \quad \mathbf{h} = \mathbf{H}(x,y,z)\,e^{nt},$$

where n is in general a complex quantity. Then from equations (2), (3), (4), (5), and (6), a complex value of n is associated with the solutions $\mathbf{U}$, Θ, Π, and $\mathbf{H}$ satisfying

$$\operatorname{div} \mathbf{U} = 0, \qquad (7)$$

$$n\frac{1}{P_1}\mathbf{U} = -Ra\,\Theta\hat{k} - \frac{1}{P_1}\operatorname{grad}\Pi - \operatorname{curl}\operatorname{curl}\mathbf{U} + Q(\operatorname{curl}\mathbf{H}) \times \hat{l}, \qquad (8)$$

$$n\Theta + \mathbf{U} \cdot \hat{k} = \nabla^2\Theta, \qquad (9)$$

$$n(P_2/P_1)\mathbf{H} - \operatorname{curl}(\mathbf{U} \times \hat{l}) = -\operatorname{curl}\operatorname{curl}\mathbf{H}, \qquad (10)$$

$$\operatorname{div}\mathbf{H} = 0, \qquad (11)$$

and the homogeneous, time-independent boundary conditions previously specified in § 3.

The complex conjugate of n, n^*, is associated with the solution set $\mathbf{U}^*$, Θ^*, Π^*, and $\mathbf{H}^*$. They satisfy the equations

$$\operatorname{div} \mathbf{U}^* = 0, \tag{12}$$

$$n^* \frac{1}{P_1} \mathbf{U}^* = -Ra\Theta^* \hat{k} - \frac{1}{P_1} \operatorname{grad} \Pi^* - \operatorname{curl} \operatorname{curl} \mathbf{U}^* + Q(\operatorname{curl} \mathbf{H}^*) \times \hat{l}, \tag{13}$$

$$n^* \Theta^* + \mathbf{U}^* \cdot \hat{k} = \nabla^2 \Theta^*, \tag{14}$$

$$n^*(P_2/P_1) \mathbf{H}^* - \operatorname{curl} (\mathbf{U}^* \times \hat{l}) = \operatorname{curl} \operatorname{curl} \mathbf{H}^*, \tag{15}$$

$$\operatorname{div} \mathbf{H}^* = 0. \tag{16}$$

The boundary conditions for $\mathbf{U}^*$, Θ^*, and $\mathbf{H}^*$ are exactly the same as those for $\mathbf{U}$, Θ, and $\mathbf{H}$.

Form the inner (dot) product of both sides of equation (8) with $\mathbf{U}^*$ and then use equation (14) to eliminate $\mathbf{U}^* \cdot \hat{k}$ from the resulting equation. After integrating the remaining terms over the region R with boundary B, one obtains

$$n \frac{1}{P_1} \int_R \mathbf{U} \cdot \mathbf{U}^* dV = -Ra \int_R \Theta \nabla^2 \Theta^* dV + Ra\, n^* \int_R \Theta\Theta^* dV - \frac{1}{P_1} \int_R \operatorname{grad} \Pi \cdot \mathbf{U}^* dV$$
$$- \int_R \operatorname{curl} \operatorname{curl} \mathbf{U} \cdot \mathbf{U}^* dV + Q \int_R (\operatorname{curl} \mathbf{H}) \times \hat{l} \cdot \mathbf{U}^* dV. \tag{17}$$

The following integral transformations resulting from application of Gauss's theorem are then used:

$$\int_R \operatorname{grad} \Pi \cdot \mathbf{U}^* dV = \oint_B \Pi \mathbf{U}^* \cdot \hat{n}\, dS - \int_R \Pi \operatorname{div} \mathbf{U}^* dV, \tag{i}$$

$$\int_R \Theta \nabla^2 \Theta^* dV = \oint_B \Theta \nabla \Theta^* \cdot \hat{n}\, dS - \int_R \nabla \Theta \cdot \nabla \Theta^* dV, \tag{ii}$$

$$\int_R \operatorname{curl} \operatorname{curl} \mathbf{U} \cdot \mathbf{U}^* dV = \oint_B (\operatorname{curl} \mathbf{U}) \times \mathbf{U}^* \cdot \hat{n}\, dS + \int_R \operatorname{curl} \mathbf{U} \cdot \operatorname{curl} \mathbf{U}^* dV. \tag{iii}$$

In equation (i), the surface integral vanishes since $\mathbf{U}^* = 0$, (B.C. 1), on all bounding surfaces. The volume integral on the right-hand side of equation (i) also vanishes since the velocity field is solenoidal throughout the region R. The surface integral in equation (iii) vanishes on all surfaces, (B.C. 1). Using the general convective-radiative exchange boundary condition, (B.C. 2), the surface integral in equation (ii) can be rewritten as

$$\oint_B \Theta \nabla \Theta^* \cdot \hat{n}\, dS = -\oint_B Bi\, \Theta\Theta^* dS.$$

Of course in the two special cases, fixed temperature on all boundaries ($\Theta = 0$) or prescribed heat-flux on all boundaries ($\nabla\Theta \cdot \hat{n} = 0$), the surface integral in equation (ii) will vanish. The remaining integral in equation (17) can be transformed as follows:

$$\int_R \operatorname{curl} \mathbf{H} \times \hat{l} \cdot \mathbf{U}^* dV = \oint_R \operatorname{curl} \mathbf{H} \cdot \hat{l} \times \mathbf{U}^* dV.$$

Using the divergence theorem one can write

$$\int_R \operatorname{curl}\mathbf{H}.\hat{l}\times\mathbf{U}^*\,dV = \oint_B \mathbf{H}\times(\hat{l}\times\mathbf{U}^*).\hat{n}\,dS + \int_R \mathbf{H}.\operatorname{curl}(\hat{l}\times\mathbf{U}^*)\,dV.$$

The surface integral vanishes since $\mathbf{U}^* = 0$, (B.C. 1). With the aid of equation (15), the volume integral on the right side is rewritten as

$$\begin{aligned}\int_R \mathbf{H}.\operatorname{curl}(\hat{l}\times\mathbf{U}^*)\,dV &= -\int_R \mathbf{H}.\operatorname{curl}(\mathbf{U}^*\times\hat{l})\,dV \\ &= -n^*\frac{P_2}{P_1}\int_R \mathbf{H}.\mathbf{H}^*\,dV - \int_R \mathbf{H}.\operatorname{curl}\operatorname{curl}\mathbf{H}^*\,dV.\end{aligned}$$

Employing the divergence theorem to transform the second integral on the right side to positive definite form, one has

$$\begin{aligned}\int_R \mathbf{H}.\operatorname{curl}(\hat{l}\times\mathbf{U}^*)\,dV &= -n^*\frac{P_2}{P_1}\int_R \mathbf{H}.\mathbf{H}^*\,dV \\ &\quad -\int_R \operatorname{curl}\mathbf{H}.\operatorname{curl}\mathbf{H}^*\,dV - \oint_B (\operatorname{curl}\mathbf{H}^*)\times\mathbf{H}.\hat{n}\,dS,\end{aligned}$$

where the surface integral vanishes, since curl $\mathbf{H}^*\times\hat{n} = 0$ (B.C. 3). The final result of these manipulations is

$$\int_R (\operatorname{curl}\mathbf{H})\times\hat{l}.\mathbf{U}^*\,dV = -n^*\frac{P_2}{P_1}\int_R \mathbf{H}.\mathbf{H}^*\,dV - \int_R \operatorname{curl}\mathbf{H}.\operatorname{curl}\mathbf{H}^*\,dV. \quad \text{(iv)}$$

Applying equations (i)–(iv) to equation (17) yields

$$(n/P_1)\,I_1 - Ra(I_2+I_3) - Ra\,n^*I_4 + I_5 + Qn^*\,(P_2/P_1)\,I_6 + QI_7 = 0, \tag{18}$$

where all I_i are real, positive definite integrals:

$$\left.\begin{aligned}&I_1 \equiv \int_R \mathbf{U}.\mathbf{U}^*\,dV, \quad I_2 \equiv \int_R \nabla\Theta.\nabla\Theta^*\,dV, \quad I_3 \equiv \oint_B Bi\Theta\Theta^*\,dS, \\ &I_4 \equiv \int_R \Theta\Theta^*\,dV, \quad I_5 \equiv \int_R \operatorname{curl}\mathbf{U}.\operatorname{curl}\mathbf{U}^*\,dV, \quad I_6 \equiv \int_R \mathbf{H}.\mathbf{H}^*\,dV, \\ &I_7 \equiv \int_R \operatorname{curl}\mathbf{H}.\operatorname{curl}\mathbf{H}^*\,dV.\end{aligned}\right\} \tag{19}$$

Separating the real and imaginary parts of equation (18), one has

$$\operatorname{Re}(n)\left\{\frac{1}{P_1}I_1 - Ra\,I_4 + Q\frac{P_2}{P_1}I_6\right\} - Ra(I_2+I_3) + I_5 + QI_7 = 0, \tag{20a}$$

$$\operatorname{Im}(n)\left\{\frac{1}{P_1}I_1 + Ra\,I_4 - Q\frac{P_2}{P_1}I_6\right\} = 0. \tag{20b}$$

Consider the following two possibilities

(1) $Ra > 0$

If $I_1 \geqslant P_2QI_6$, then $Ra > 0$ implies that $\operatorname{Im}(n) = 0$ from equation (20b). Hence in this case n must be real. Of course, if $I_1 < P_2QI_6$, then one cannot conclude that the imaginary part of n will vanish.

(2) $Ra < 0$

This implies that $\mathrm{Re}\,(n) < 0$ from equation (20a). Thus if $I_1 \geqslant P_2 Q I_6$ and n is to have an imaginary part, then the imaginary part must be associated with $\mathrm{Re}\,(n) < 0$, a decaying perturbation. Instabilities ($\mathrm{Re}\,(n) > 0$) are associated with real values of n. The limiting case for the onset of motion is now given by $\mathrm{Re}\,(n) = n = 0$. Thus the transition from stability to instability occurs through a stationary state, and the principle of exchange of stabilities is established. A sufficient condition for the exchange principle to be valid is

$$I_1 \geqslant P_2 Q I_6,$$

or in dimensional form

$$\rho \int_R \mathbf{U}.\mathbf{U}^* \, dV \geqslant \frac{\mu}{4\pi} \int_R \mathbf{H}.\mathbf{H}^* \, dV. \tag{21}$$

Physically, this is equivalent to specifying that the total kinetic energy associated with a disturbance is greater than, or equal to, its total magnetic energy. Chandrasekhar (1952) conjectured the above sufficiency condition, but he did not present a rigorous proof of the supposition as has been developed herein.

If no external magnetic field is applied to the fluid, then $Q = 0$ in equations (20a) and (20b). Hence one recovers a generalization of the Pellew & Southwell proof of the exchange principle for horizontal layers. Here the exchange principle is established for arbitrary regions in which the fluid is initially heated from below.

For this case ($Q = 0$) also a planar, free-surface can replace a rigid one. At such a surface, (B.C. 4) is utilized in equations (i) and (iii) instead of (B.C. 1) and the proof follows directly.

5. The case of very large Q

The sufficiency condition given by equation (21) to establish the exchange principle is of limited value since one cannot *a priori* be certain when this condition will be satisfied. It would be more useful to express the sufficiency condition in terms of fluid properties alone. Thompson and Chandrasekhar indicated this type of condition (equation (1)) but, as has been previously discussed, they studied the special horizontal layer geometry with a vertical, applied magnetic field. Moreover, a very artificial boundary condition must be maintained on the induced magnetic field at the horizontal bounding planes.

We now consider the problem of finding a sufficiency condition related to quantities that are known at the beginning. If one operates on equation (7) with $1/Q\,\{n(P_2/P_1) - \nabla^2\}$, where of course $\nabla^2 \equiv \mathrm{grad\,div} - \mathrm{curl\,curl}$, then one obtains

$$\begin{aligned}\frac{n}{QP_1}\left(n\frac{P_2}{P_1} + \mathrm{curl\,curl}\right)\mathbf{U} - \frac{Ra}{Q}\left(n\frac{P_2}{P_1} - \nabla^2\right)\Theta\hat{k} \\ + \frac{1}{QP_1}\left(n\frac{P_2}{P_1} - \mathrm{grad\,div}\right)\mathrm{grad}\,\Pi - n\frac{P_2}{P_1}(\mathrm{curl}\,\mathbf{H}) \times \hat{l} \\ + \frac{n}{Q}\frac{P_2}{P_1}\mathrm{curl\,curl}\,\mathbf{U} + \hat{l} \times \mathrm{curl\,curl\,curl}\,\mathbf{H} = -\frac{1}{Q}\mathrm{curl\,curl\,curl\,curl}\,\mathbf{U}.\end{aligned} \tag{22}$$

The solenoidal properties of the velocity $\mathbf{U}$ and of the induced magnetic field $\mathbf{H}$ have been used to derive equation (22). Multiply equation (22) by $\mathbf{U}^*$ and use equation (14) to eliminate $\mathbf{U}^*.\hat{k}$ from the resulting equation. The remaining terms are then integrated over the region R.

$$\frac{n^2P_2}{QP_1^2}\int_R \mathbf{U}.\mathbf{U}^*dV+\frac{n}{QP_1}(1+P_2)\int_R \operatorname{curl}\operatorname{curl}\mathbf{U}.\mathbf{U}^*dV-\frac{Ra}{Q}$$
$$\times\left\{nn^*\frac{P_2}{P_1}\int_R \Theta\Theta^*dV-n\frac{P_2}{P_1}\int_R \Theta\nabla^2\Theta^*dV-n^*\int_R \Theta^*\nabla^2\Theta\,dV+\int_R \nabla^2\Theta\nabla^2\Theta^*dV\right\}$$
$$+\frac{1}{QP_1}\int_R\left\{\left(n\frac{P_2}{P_1}-\operatorname{grad}\operatorname{div}\right)\operatorname{grad}\Pi\right\}.\mathbf{U}^*dV-n\frac{P_2}{P_1}\int_R(\operatorname{curl}\mathbf{H})\times\hat{l}.\mathbf{U}^*dV$$
$$+\int_R \hat{l}\times\operatorname{curl}\operatorname{curl}\operatorname{curl}\mathbf{H}.\mathbf{U}^*dV=-\frac{1}{Q}\int_R\operatorname{curl}\operatorname{curl}\operatorname{curl}\operatorname{curl}\mathbf{U}.\mathbf{U}^*dV. \qquad (23)$$

To obtain some insight into this problem, consideration will be given to the limiting case of very large Q (the ratio of magnetic to viscous forces). The effect of viscosity is thus significant near the bounding surfaces and in the above integrated equation, the integral on the right side (which results from the viscous force) is negligible in comparison with the last integral on the left side (resulting from the magnetic force). Of course the presence of viscosity will require $\mathbf{u}=0$, (B.C. 1), at the rigid bounding surfaces; if these surfaces are much better electrical conductors than the fluid, $\hat{n}\times\operatorname{curl}\mathbf{h}=0$, (B.C. 3).

Thus we now work with equation (23) where the right side is equal to zero. It has been previously indicated how, upon application of Gauss' theorem, the second, fourth, fifth, and eighth integrals may be transformed to positive definite form. In addition, by employing Gauss's theorem in conjunction with solenoidal property of the velocity field, the seventh integral can be shown to vanish. The last integral is treated as follows:

$$\int_R \hat{l}\times\operatorname{curl}\operatorname{curl}\operatorname{curl}\mathbf{H}.\mathbf{U}^*dV=\int_R\operatorname{curl}\operatorname{curl}\operatorname{curl}\mathbf{H}.\mathbf{U}^*\times\hat{l}\,dV$$
$$=\oint_B(\operatorname{curl}\operatorname{curl}\mathbf{H})\times(\mathbf{U}^*\times\hat{l}).\hat{n}\,dS+\int_R\operatorname{curl}\operatorname{curl}\mathbf{H}.\operatorname{curl}(\mathbf{U}^*\times\hat{l})\,dV.$$

The surface integral vanishes since $\mathbf{U}^*=0$ on all boundaries, (B.C. 1). The volume integral is rewritten with the aid of equation (15), thus

$$\int_R \hat{l}\times\operatorname{curl}\operatorname{curl}\operatorname{curl}\mathbf{H}.\mathbf{U}^*dV=\int_R\operatorname{curl}\operatorname{curl}\mathbf{H}.\left\{n^*\frac{P_2}{P_1}\mathbf{H}^*+\operatorname{curl}\operatorname{curl}\mathbf{H}^*\right\}dV$$
$$=n^*\frac{P_2}{P_1}\int_R\operatorname{curl}\mathbf{H}.\operatorname{curl}\mathbf{H}^*dV+\int_R\operatorname{curl}\operatorname{curl}\mathbf{H}.\operatorname{curl}\operatorname{curl}\mathbf{H}^*dV, \qquad (\mathrm{v})$$

where Gauss's theorem and (B.C. 3) have been used to make the last transformation. After applying the above results to the integral equation (23) (with the right side equal to zero), one obtains

$$\frac{n^2}{Q}\frac{P_2}{P_1^2}I_1+\frac{n}{QP_1}(1+P_2)I_5-\frac{Ra}{Q}\left\{\left(\frac{P_2}{P_1}n+n^*\right)(I_2+I_3)+nn^*\frac{P_2}{P_1}I_4+I_8\right\}$$
$$+nn^*\frac{P_2^2}{P_1^2}I_6+\frac{P_2}{P_1}(n+n^*)I_7+I_9=0, \qquad (24)$$

where I_1 through I_7 have been previously defined (see equation (19)) and

$$I_8 \equiv \int_R \nabla^2\Theta \cdot \nabla^2\Theta^* \, dV, \quad I_9 \equiv \int_R \operatorname{curl}\operatorname{curl}\mathbf{H} \cdot \operatorname{curl}\operatorname{curl}\mathbf{H}^* \, dV. \tag{25}$$

Taking the imaginary part of equation (24) yields

$$\operatorname{Im}(n)\left[2\operatorname{Re}(n)\, I_1 + \frac{1}{QP_1}(1+P_2)\, I_5 - \frac{Ra}{Q}\left(\frac{P_2}{P_1}-1\right)(I_2+I_3)\right] = 0. \tag{26}$$

Instabilities ($\operatorname{Re}(n) > 0$) are associated with $Ra > 0$. Thus from equation (26) one concludes that a sufficient condition for $\operatorname{Im}(n) = 0$ when $\operatorname{Re}(n) > 0$ is

$$P_2/P_1 \leqslant 1 \quad \text{or} \quad k \leqslant \eta.$$

This is precisely the sufficient condition proposed by Thompson (1951) and Chandrasekhar (1952) to establish the exchange principle for the special cases they each treated. In this section the result has been established for arbitrary configurations and for arbitrary direction of the applied magnetic field. The result is limited to situations where the magnetoviscous parameter, Q, is very large.

6. Concluding remarks

It should be pointed out that the proof of the exchange principle does not lead to the removal of the unsteady terms in all stability problems. For example, the Rayleigh–Taylor problem (i.e. the stability of a heterogeneous fluid layer with density and viscosity gradients parallel to the body force which is perpendicular to the fluid layer) is somewhat analogous to the thermal stability problem in that the initial configurations are seemingly identical. However, even though the exchange principle has been proven for that problem (see Chandrasekhar 1955 and Selig 1964), the unsteady terms are always retained in its analysis (i.e. $\operatorname{Re}(n)$ is not set equal to zero). Sherman (1965) shows that it is not possible to obtain information about the marginal state of the Rayleigh–Taylor problem by removing the unsteady terms.

The essential physical difference between the Rayleigh–Taylor and the thermal-stability problems is that in the former the adverse density gradient which promotes instability is destroyed by the motion (induced by the instability), i.e. the state after the instability sets in is a rest state. In the latter case, the heating from below maintains the unstable density gradient so that a maintained convective flow (the Bénard cells) follows the instability (see Sherman 1965 for more details). It thus appears that if conditions promoting the instability are removed by the action of the instability, then the unsteady terms must be retained in the analysis even if it can be shown that the principle of exchange of stabilities is valid for the case under consideration.

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